ETH
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Swiss Federal Institute of Technology Zurich

IFOR

Determining Optimal Control Policies for Supply Networks Under Uncertainty

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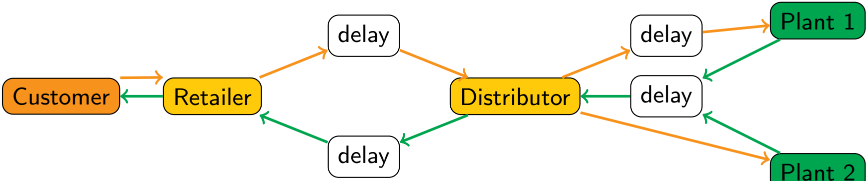
Problem Setting

Given: Supply network



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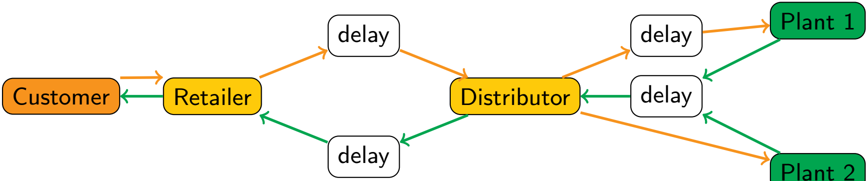

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Links : Flows

- Information flows (orders)
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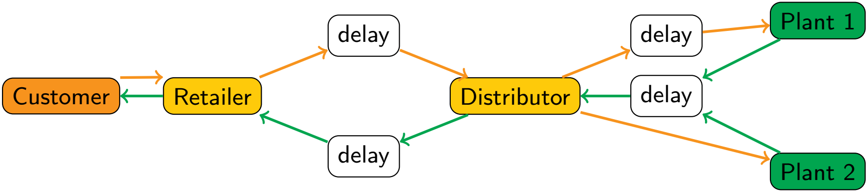

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Links : Flows
- Information flows (orders)
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Assumption : Material flow to be coordinated via information flow
Goal : Determine optimal control policy

Overview

1. Supply Networks as Controlled Discrete-time Systems
2. Stochastic Robust Optimal Control
3. Numerical Example

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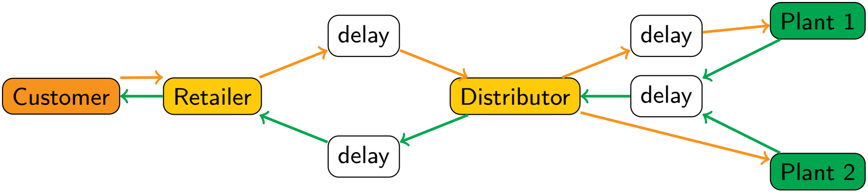
Discrete-time Supply Network Model



Nodes $v \in V$: Places where material is stored

Edges $e \in E$: Information and material flow

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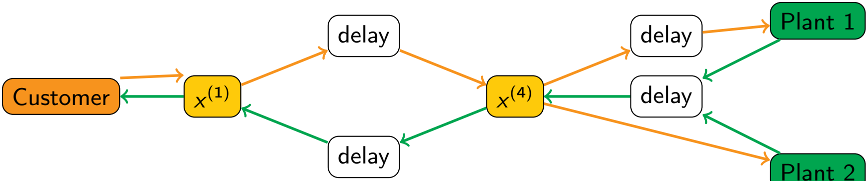

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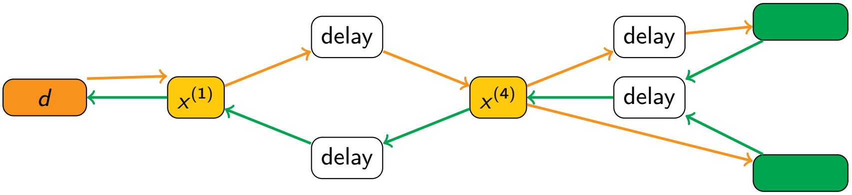

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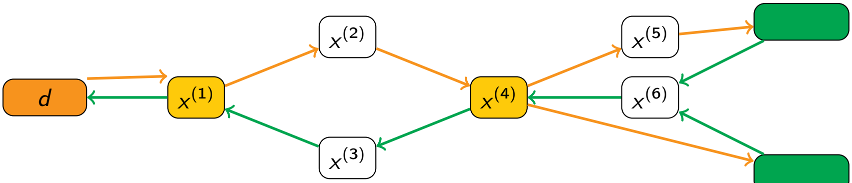

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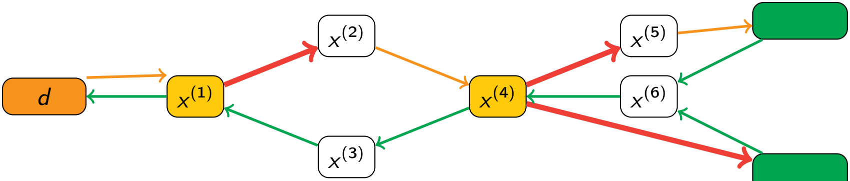

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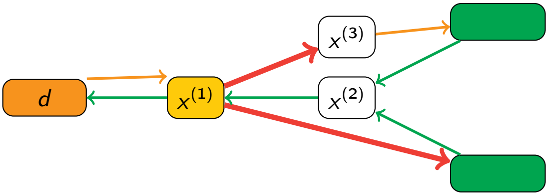
Control: Orders $u^{(i)}(x)$ of facilities as function of state x

Example



$$x_{k+1}^{(1)} = x_k^{(1)} + x_k^{(2)} - d_k \tag{1}$$

$$x_{k+1}^{(2)} = x_k^{(3)} + u_k^{(1)} \tag{2}$$

$$x_{k+1}^{(3)} = u_k^{(2)} \tag{3}$$

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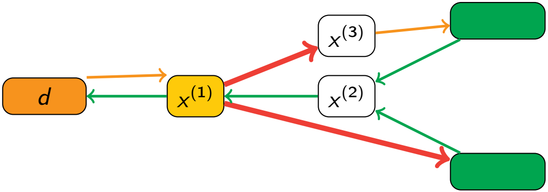

$$x_{k+1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x_k + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u_k + d_k$$

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Finite Horizon Robust Optimal Control

Given

- Dynamics $x_{k+1} = Ax_k + Bu_k + d_k$
- Linear constraints on state and controls $Fx + Gu \leq g$
- Linear costs on state ($p^T x$) and control ($q^T u$)
- Stochastic disturbances $d_k \in \mathcal{D} = \mathrm{conv}\{d^1, d^2, \ldots, d^\ell\}$
- Fixed time horizon N

Let $u_k(x_k)$ denote the control input when system is in state x at time k time steps to go. Let $\pi = (u_1, \ldots, u_N)$.

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$$\hat{J}(x_1, \pi) := \max_{(d_1, \ldots, d_{N-1})} \sum_{i=1}^{N-1} [p^T x_i + q^T u_i(x_i)] + p^T x_N$$

where $x_{k+1} = Ax_k + Bu_k(x_k) + d_k$.

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$$\bar{J}(x_1, \pi) := \mathbb{E}\left(\sum_{i=1}^{N-1}(p^T x_i + q^T u_i(x_i)) + p^T x_N\right)$$

where $x_{k+1} = Ax_k + Bu_k(x_k) + Ed_k$.

Stochastic Dynamic Programming

Let $J_k^* : \mathcal{X} \longrightarrow \mathbb{R}$ the optimal value function at time k .

Terminal condition: $J_N^*(x) = p_N^T x$ and $\mathcal{X}_N = \{x \in \mathbb{R}^n : H_N x \leq h_N\}$.

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(Then, J_k^* and $\mathcal{X}_k$ are so, too.)

Properties

Theorem

- $J_k^*(X)$ *is piecewise affine and convex in* x*.*
- $x_k^*(X)$ *is piecewise affine and continuous in* x*.*

$$\begin{cases} J_k^*(x) = V_k^{(i)} x + W_k^{(i)} \\ u_k^*(x) = R_k^{(i)} x + S_k^{(i)} \end{cases} \quad \textit{for } x \in \mathcal{R}_k^{(i)}.$$

where $\mathcal{R} = \{\mathcal{R}_k^{(i)}\}$ *is a partition of* $\mathcal{X}_k$ *with* $\mathrm{cl}(\mathcal{R}_k^{(i)})$ *being polyhedra.*

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Goal: We would like to approximate $u^* = \lim_{N\to\infty} u_1^*$.

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Example: One Retailer With Two Suppliers

$$x_{k+1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x_k + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u_k + d_k, .$$

u_1: orders at supplier 1 with lead time 1 and unit cost 4
u_2: orders at supplier 2 with lead time 2 and unit cost 1
$u_1, u_2 \leq 8$

Resulting optimal control law: (w.r.t. worst-case cost if $d[t] \in [0, 8]$)

$$u_1^*(x) = \min\{\max\{20 - x_1 - x_2 - x_3 - x_4, 0\}, 4\},$$
$$u_2^*(x) = \max\{16 - x_1 - x_2 - x_3 - x_4, 0\}.$$

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$$u_1^*(x) = \min\{\max\{22 - x_1 - x_2 - x_3 - x_4, 0\}, 4\},$$
$$u_2^*(x) = \max\{16 - x_1 - x_2 - x_3 - x_4, 0\}.$$

Conclusions

- Able to compute explicit state-feedback control policies for general networks if
  - the dynamics of the nodes is linear
  - cost and constraints are piecewise linear and convex
- Current work:
  - Consider approximate parametric LPs solvers (solution can lead to exponential number of regions)
  - Decentralized control: coordination via 'parametric' contracs