SFB 637
Teilprojekt A5

Application of small gain type theorems in logistics of autonomous processes

Sergey Dashkovskiy, Björn Rüffer, Fabian Wirth

Zentrum für Technomathematik

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Motivation

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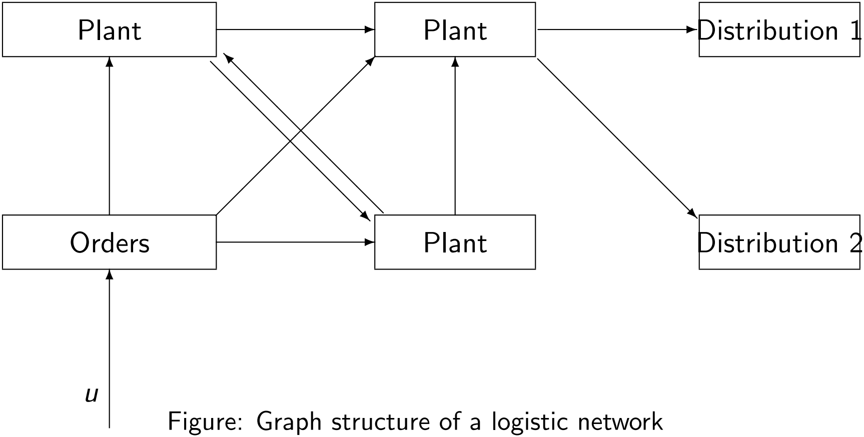
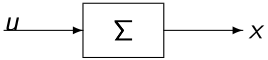


Figure: Graph structure of a logistic network

Autonomous dynamical node



$\Sigma: \quad \dot{x} = f(x, u)$

u : input

x : state

input-to-state stability $\sim$

the level of x is proportional

to the level of u

Autonomous dynamical node

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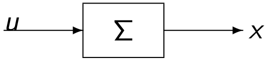

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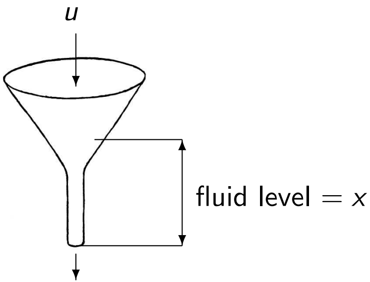

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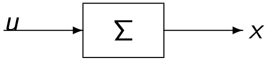

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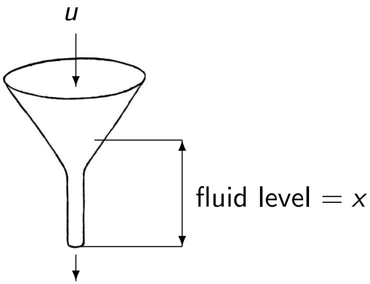

stability $\Rightarrow$ diameter $d = g(x)$

Autonomous dynamical node

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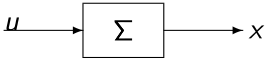

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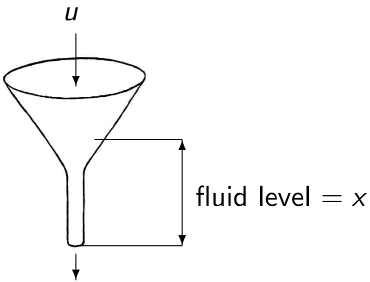


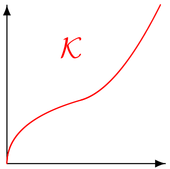
stability $\Rightarrow$ diameter $d = g(x)$

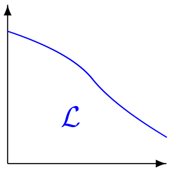
$=$ autonomous control

Stability of singular node

A system $\dot{x}(t) = f(x(t), u(t))$ is called stable, if

$$||x(t)|| \leq \beta(||x(0)||, t) + \gamma(||u||_\infty).$$





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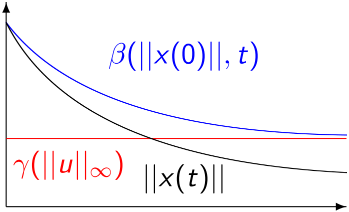


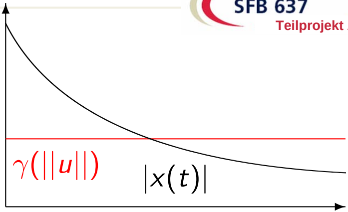
ISS-Lyapunov function

V is called ISS-Lyapunov function for

$$\dot{x}(t) = f(x(t), u(t))$$

if

$$|x(t)| \geq \gamma(||u||) \Rightarrow \nabla V(x) f(x, u) < 0.$$



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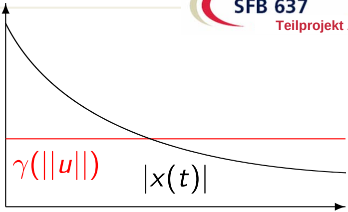

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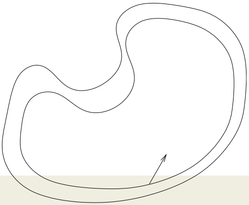
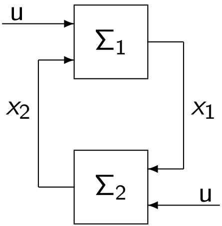


Feedback of two systems

Consider

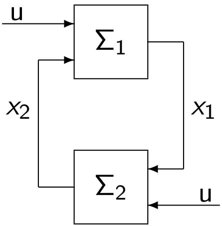
$$\begin{aligned}\Sigma_1 : \dot{x}_1 &= f_1(x_1, x_2, u)\\ \Sigma_2 : \dot{x}_2 &= f_2(x_1, x_2, u)\end{aligned}$$



Feedback of two systems

Consider

$$\Sigma_1 : \dot{x}_1 = f_1(x_1, x_2, u)$$
$$\Sigma_2 : \dot{x}_2 = f_2(x_1, x_2, u)$$



s.t. each subsystem is stable and has a Lyapunov function V_i

Feedback of two systems

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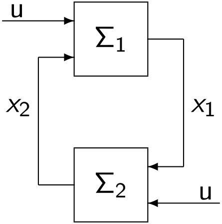


s.t. each subsystem is stable and has a Lyapunov function V_i

$$|x_i| \geq \max\{\gamma_{ij}(|x_j|), \gamma(|u|)\} \Rightarrow \nabla V_i f_i(x_1, x_2, u) < 0.$$

Stability criterion for two systems

Theorem (Jiang, Mareels, Wang 1996)

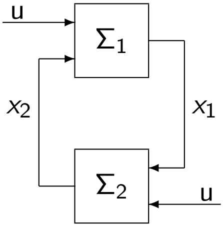
If

$$\gamma_{12} \circ \gamma_{21} \leq \mathrm{Id},$$

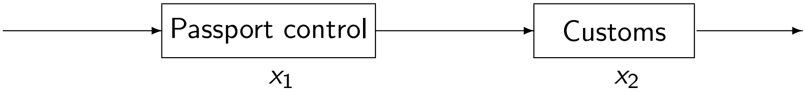
then

$$\dot{x} = f(x, u)$$

is stable.



An example with two nodes



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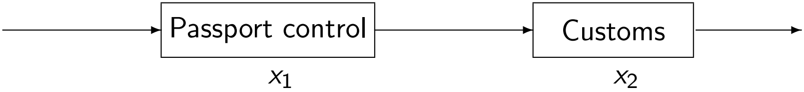

$$\dot{x}_1 = u - b_1$$
$$\dot{x}_2 = b_1 - b_2$$

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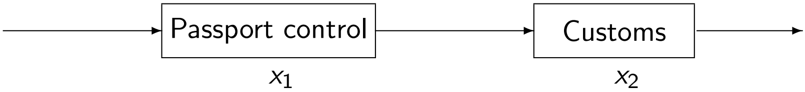

$$\dot{x}_1 = u - b_1$$

$$\dot{x}_2 = b_1 - b_2$$

$$b_1 = \frac{a x_1 + b\sqrt{x_1}}{1 + x_2}$$

$$b_2 = c x_2$$

Two nodes

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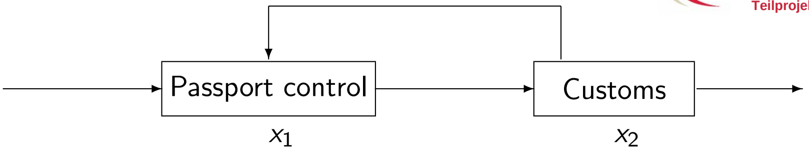
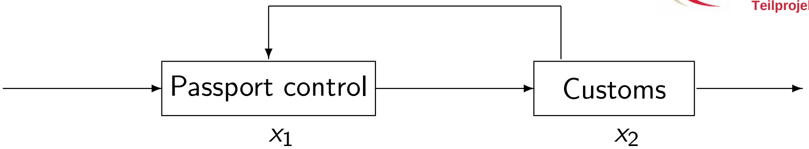
Teilprojekt A5



$$\dot{x}_1 = u - \frac{ax_1 + b\sqrt{x_1}}{1 + x_2}$$
$$\dot{x}_2 = \frac{ax_1 + b\sqrt{x_1}}{1 + x_2} - cx_2$$

Two nodes



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$$\gamma_u(u) = \frac{b^2}{4} u^2$$
$$\gamma_{12}(x_2) = \frac{a^2}{b^2} x_2^2$$
$$\gamma_{21}(x_1) = \left(\min \left\{ \frac{c}{2a}, \frac{c^2}{b^2} \right\} \right)^{\frac{1}{2}} \sqrt{x_1}$$

Two nodes

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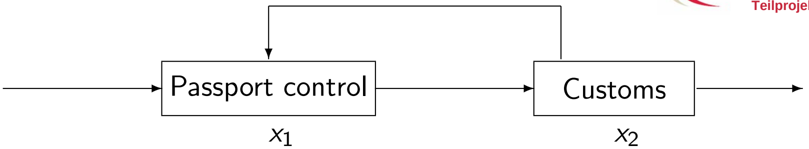

$$\dot{x}_1 = u - \frac{ax_1 + b\sqrt{x_1}}{1+x_2}$$
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$$\gamma_{12} \circ \gamma_{21}(s) = \frac{a^2}{b^2}\min\left\{\frac{c}{2a}, \frac{c^2}{b^2}\right\}s \leq \frac{a^2c^2}{b^4}s < s$$
if $\quad \frac{ac}{b^2} < 1.$

Simulation

$$u(t) = 1.5 + \sin(t) + \sin(5t)/2$$
$$x_1(0) = 10, \quad x_2(0) = 10.$$

Simulation

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$$u(t) = 1.5 + \sin(t) + \sin(5t)/2$$
$$x_1(0) = 10, \quad x_2(0) = 10.$$

$$\dot{x}_1(t) = u(t) - \frac{x_1(t)+\sqrt{x_1(t)}}{1+x_2(t)}$$
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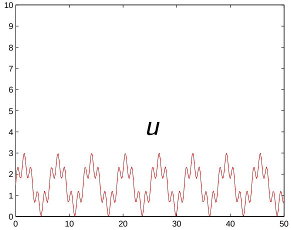

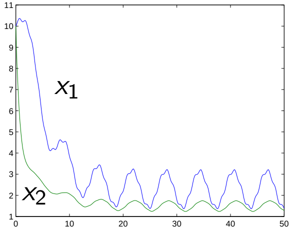

Figure: Input u and the queues x_1, x_2

n coupled systems

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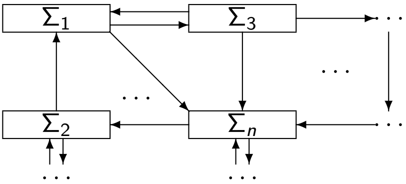
Consider

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, \ldots, x_n, u) \\ &\vdots \\ \dot{x}_n &= f_n(x_1, \ldots, x_n, u) \end{aligned}$$



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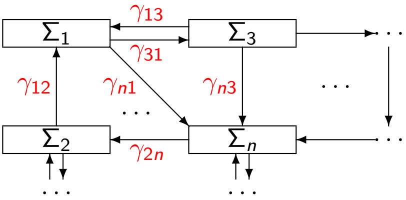

$$\Gamma := (\gamma_{ij}) = \begin{bmatrix} 0 & \gamma_{12} & \dots & \dots & \gamma_{1n} \\ \gamma_{21} & 0 & \gamma_{23} & \dots & \gamma_{2n} \\ \vdots & & & & \vdots \\ \gamma_{n-1,1} & \dots & \gamma_{n-1,n-2} & 0 & \gamma_{n-1,n} \\ \gamma_{n1} & \dots & \dots & \gamma_{n,n-1} & 0 \end{bmatrix}$$

n coupled systems

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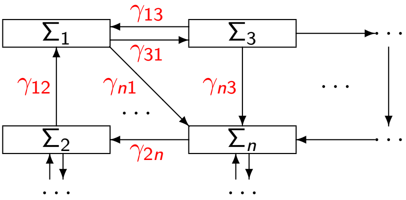

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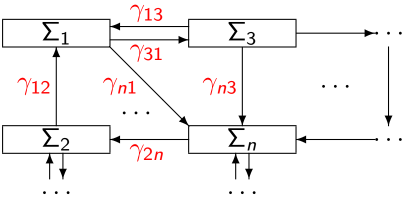
When such a network is stable?

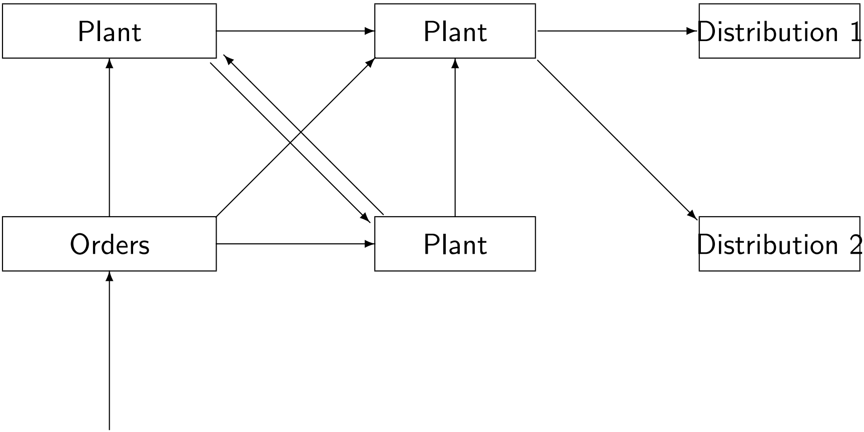
Stability criterion

Theorem

Let the small gain condition holds true $\Gamma(s) \not\geq s$, *for all vectors* s *with nonnegative components. Then the overall system* $\dot{x} = f(x, u)$ *is stable.*



Car production network



Car production network

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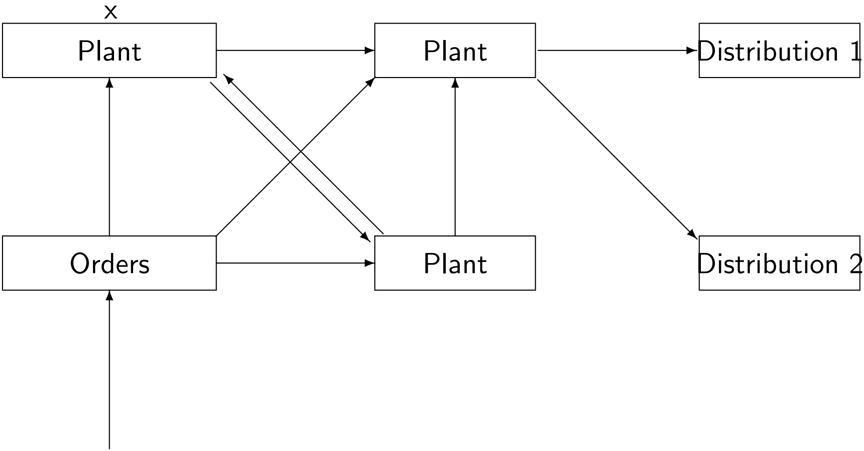

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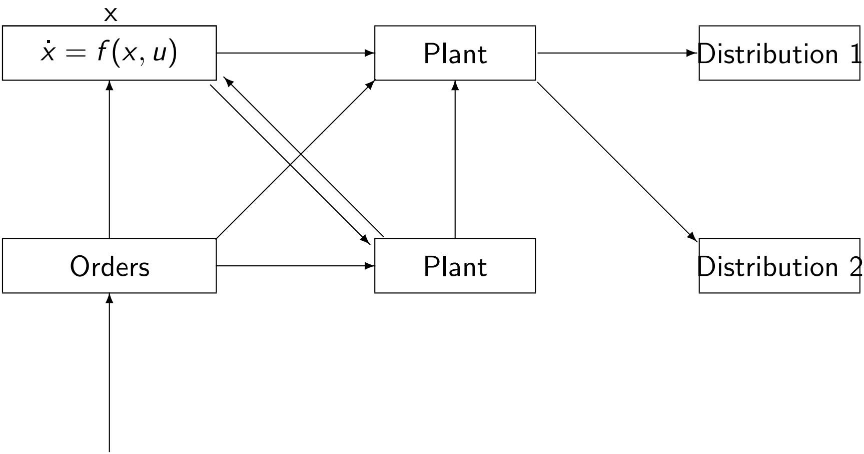

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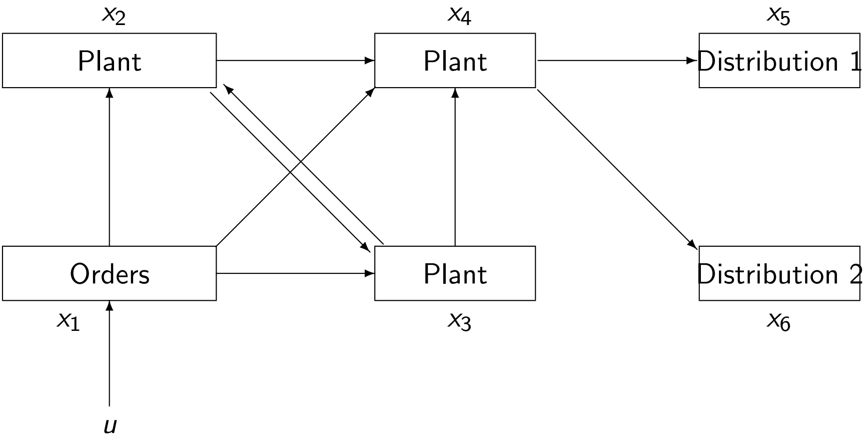

State equations

$$\dot{x}_1 = u - \frac{ax_1 + b\sqrt{x_1}}{1 + x_2 + x_3}$$

$$\dot{x}_2 = \frac{1}{3}\frac{ax_1 + b\sqrt{x_1}}{1 + x_2 + x_3} + \frac{1}{2}\min\{b_3, c_3x_3\} - \min\{b_2, c_2x_2\}$$

$$\dot{x}_3 = \frac{1}{3}\frac{ax_1 + b\sqrt{x_1}}{1 + x_2 + x_3} + \frac{1}{2}\min\{b_2, c_2x_2\} - \min\{b_3, c_3x_3\}$$

$$\dot{x}_4 = \frac{1}{3}\frac{ax_1 + b\sqrt{x_1}}{1 + x_2 + x_3} + \frac{1}{2}\min\{b_2, c_2x_2\} + \min\{b_3, c_3x_3\} - \min\{b_4, c_4x_4\}$$

$$\dot{x}_5 = \frac{1}{2}\min\{b_4, c_4x_4\} - c_5x_5$$

$$\dot{x}_6 = \frac{1}{2}\min\{b_4, c_4x_4\} - c_6x_6$$

Gain-Matrix

$$\Gamma := (\gamma_{ij}) = \begin{bmatrix} 0 & a_{12}x^2 & a_{13}x^2 & 0 & 0 & 0 \\ a_{21}\sqrt{x} & 0 & a_{23}x & 0 & 0 & 0 \\ a_{31}\sqrt{x} & a_{32}x & 0 & 0 & 0 & 0 \\ a_{41}\sqrt{x} & a_{42}x & a_{43}x & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{54}x & 0 & 0 \\ 0 & 0 & 0 & a_{64}x & 0 & 0 \end{bmatrix}$$

Gain-Matrix

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small-gain condition:

$$\Gamma(s) = \begin{bmatrix} a_{12}s_2^2 + a_{13}s_3^2 \\ a_{21}\sqrt{s_1} + a_{23}s_3 \\ a_{31}\sqrt{s_1} + a_{32}s_2 \\ a_{41}\sqrt{s_1} + a_{42}s_2 + a_{43}s_3 \\ a_{54}s_4 \\ a_{64}s_4 \end{bmatrix} \not\geq \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \end{bmatrix}$$

Conclusions

- Stability is one of the basic properties in dynamics of logistic processes
- We have presented a stability criterion for logistic networks
- This criterion can be applied for networks of autonomous systems
- We have considered examples to show how this criterion can be applied in reality
- This criterion can be used for design of logistic networks